An Efficient Technique for Computing the Potential Green's Functions for a Thin, Periodically Excited Parallel-Plate Waveguide Bounded by Electric and Magnetic Walls

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Abstract —An *efficient* formulation is described for calculating all vector and scalar potential Green's functions for a thin, infinitely long waveguide with periodic excitation. The Green's functions are represented by the first few terms of the modal expansion plus a quasi-static correction. This allows one to compute the Green's functions over a wide band of frequencies with little additional effort over that required for a single frequency. An attractive feature of the method is that the $1/R$ free-space singularity exhibited by the potentials is explicitly extracted in the lowest order quasi-static term. This is convenient for evaluating method-of-moments self-term contributions in closed form. The Green's functions have application for problems involving stripline structures such as Rotman lenses.

I. Introduction

INFINITE-ARRAY theory [1], in combination with the method of moments [2], provides a powerful means of analyzing large periodic structures. For instance, Pozar and Schaubert [3] used this technique to study an array of microstrip antennas. The difficulty with this method is that one must have available an *efficient* means to compute the infinite-array Green's function needed in the moment-method procedure. In this paper, we describe how to efficiently compute the scalar and vector potential Green's functions for a periodically excited waveguide bounded by magnetic and electric walls. This particular application arose from consideration of a stripline Rotman lens structure, to be described in a later communication.

The acceleration of the Green's functions involves the following steps. First, the Green's functions are expanded into a spectral series. Second, the contributions due to all but the first few resonant modes are accounted for by three quasi-static terms, the lowest order of which contains the source singularity. This decomposition into quasi-static and dynamic terms is similar in principle to the technique described in [4]. The resulting expression then contains a few (possibly zero) modes and a second-degree polynomial in the frequency squared representing the quasi-static contribution. Once the coefficients of this polynomial are determined, the evaluation of the Green's function becomes trivial, so that analysis over a wide band of frequencies becomes efficient. Third, the coefficients of the quasi-static terms are represented by very slowly converging spectral series. These series have been highly accelerated, making their evaluation also efficient.

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II. Formulation of the Green's Functions

A unit cell of the periodic structure under consideration is depicted in Fig. 1. The rectangular waveguide is bounded at $z=0$, $z=a$, and $y=h$ by perfect electric conducting (PEC) walls and at $y=0$ by a perfect magnetic conducting (PMC) wall, while the Floquet walls at $|x|=w/2$ bound the extent of a single period in x. The excitation is taken to be uniform in amplitude, with linear phase variation in the x direction of the form $e^{j\beta x}$ (assuming an $e^{j\omega t}$ time convention). Thus, any rectangular field component U must satisfy the periodicity condition

$$U(w/2, y, z) = U(-w/2, y, z)e^{j\beta w} \tag{1}$$

where w is the x dimension of the unit cell. This forms what is, in effect, a cavity. The cavity occupies volume $V = ahw$ and is filled with homogeneous dielectric material of permittivity ϵ and permeability μ. The wavenumber in the dielectric is k.

The various potentials are related to the electric and magnetic fields in the cavity via

$$\bar{E} = -j\omega\bar{A} - \nabla\Phi + \frac{1}{\epsilon}\nabla\times\bar{F} \tag{2}$$

$$\bar{H} = j\omega\bar{F} - \nabla\Psi + \frac{1}{\mu}\nabla\times\bar{A}. \tag{3}$$

In the above equations, $\bar{A}$ is the magnetic vector potential, Φ the electric scalar potential, $\bar{F}$ the electric vector potential, and Ψ the magnetic scalar potential. The potentials are taken to satisfy the Lorentz gauge, so

$$(\nabla^2 + k^2)\bar{A} = -\mu\bar{J} \tag{4}$$

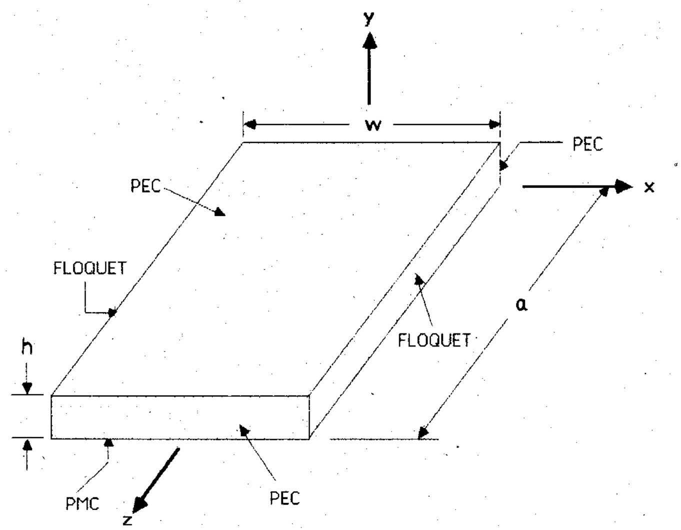

Fig. 1. The unit Floquet cell showing the periodic phase shift (Floquet) walls at $|x| = w/2$ and perfect electric and magnetic conducting walls at the remaining boundaries. This structure is derived from that of a stripline with ground plane spacing $b = 2h$ by considering the symmetry of the fields about $y = 0$.

and

$$(\nabla^2 + k^2)\Phi = -\frac{\rho}{\epsilon} \tag{5}$$

$$(\nabla^2 + k^2)\bar{F} = \epsilon\bar{M} \tag{6}$$

$$(\nabla^2 + k^2)\Psi = -\frac{\rho_m}{\mu} \tag{7}$$

where $\bar{J}$ and ρ are the electric current and charge densities, respectively, and $\bar{M}$ and ρ_m are the corresponding magnetic quantities.

By a straightforward extension of [5], we may take the boundary conditions to be as stated in Table I. The Green's functions can be expanded as series of eigenfunctions using standard techniques [6]. After some algebraic manipulations, the following expressions for the Green's functions are obtained:

$$A_x = \frac{\mu}{2V}(\Sigma_1 - \Sigma_2 - \Sigma_3 + \Sigma_4) \tag{8}$$

$$A_y = \frac{\mu}{2V}(-\Sigma_1 + \Sigma_2 - \Sigma_3 + \Sigma_4) \tag{9}$$

$$A_z = \frac{\mu}{2V}(-\Sigma_1 - \Sigma_2 + \Sigma_3 + \Sigma_4) \tag{10}$$

$$F_x = \frac{\epsilon}{2V}(-\Sigma_1 - \Sigma_2 - \Sigma_3 - \Sigma_4) \tag{11}$$

$$F_y = \frac{\epsilon}{2V}(\Sigma_1 + \Sigma_2 - \Sigma_3 - \Sigma_4) \tag{12}$$

$$F_z = \frac{\epsilon}{2V}(\Sigma_1 - \Sigma_2 + \Sigma_3 - \Sigma_4) \tag{13}$$

$$\Phi = \frac{1}{2\epsilon V}(\Sigma_1 - \Sigma_2 - \Sigma_3 + \Sigma_4) \tag{14}$$

$$\Psi = \frac{1}{2\mu V}(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4). \tag{15}$$

Note that the symbols used on the left-hand sides of

TABLE I
BOUNDARY CONDITIONS ON POTENTIALS

PEC Wall	PMC Wall
$\Phi = 0$	$\Psi = 0$
$\frac{\partial \Psi}{\partial n} = 0$	$\frac{\partial \Phi}{\partial n} = 0$
$\hat{n} \times \bar{A} = \bar{0}$	$\hat{n} \times \bar{F} = \bar{0}$
$\nabla_n \cdot \bar{A} = 0$	$\nabla_n \cdot \bar{F} = 0$
$\hat{n} \times (\nabla \times \bar{F}) = \bar{0}$	$\hat{n} \times (\nabla \times \bar{A}) = \bar{0}$
$\hat{n} \cdot \bar{F} = 0$	$\hat{n} \cdot \bar{A} = 0$

(8)–(15) have now been redefined to represent potential Green's functions rather than the actual potentials. The modal series Σ_s is defined for $s = 1, 2, 3$, and 4 by

$$\Sigma_s = \sum_{n=0}^{\infty} \cos(k_y y_s) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{e^{j(k_x x_s + k_z z_s)}}{k_{mnl}^2 - k^2} \tag{16}$$

where

$$k_y(n) = \frac{(n+\frac{1}{2})\pi}{h} \quad k_x(m) = \frac{2m\pi}{w} + \beta \quad k_z(l) = \frac{l\pi}{a} \tag{17a}$$

$$k_{mnl}^2 = k_x^2 + k_y^2 + k_z^2 \tag{17b}$$

$$x_1 = x_2 = x_3 = x_4 = x - x' \tag{18}$$

$$y_1 = y_2 = 2h - (y + y') \quad y_3 = y_4 = y' - y \tag{19}$$

$$z_1 = z_3 = z + z' \quad z_2 = z_4 = z - z'. \tag{20}$$

The field observation point $\bar{r} = (x, y, z)$ and the source point $\bar{r}' = (x', y', z')$, are both within or on the boundary of the rectangular cavity. The problem of calculating the Green's functions has now been reduced to efficiently evaluating Σ_s for $s = 1,2,3,4$. This task is addressed next.

The series in (16) is slowly convergent and unsuitable for numerical evaluation. Our strategy for accelerating (16) is essentially a Kummer's transformation [7]. The summand is approximated by a smooth function possessing a Fourier transform in the summation indices. This "regular asymptotic equivalent" is then accelerated by means of the Poisson summation formula [8].

Let $N = \{0,1,2,\cdots\}$, $\tilde{N} = \{\cdots,-2,-1,0,1,2,\cdots\}$, and $k_{\max}$ be the wavenumber k in the cavity at the highest frequency of interest. For $\xi \geq 0$, define

$$\mathscr{A}_\xi = \{(m,n,l) \in \tilde{N} \times N \times \tilde{N}: k_{mnl} > \xi k_{\max}\} \tag{21}$$

$$\bar{\mathscr{A}}_\xi = \mathscr{A}_0 - \mathscr{A}_\xi = \{(m,n,l) \in \tilde{N} \times N \times \tilde{N}: k_{mnl} \leq \xi k_{\max}\}. \tag{22}$$

(Note that $\mathscr{A}_0$ is just the set of all mode indices.) Thus, our choice of ξ partitions the three-dimensional summation lattice $\tilde{N} \times N \times \tilde{N}$ into a finite ($\bar{\mathscr{A}}_\xi$) and an infinite ($\mathscr{A}_\xi$) region. If ξ is chosen suitably larger than 1, then the factor $1/(k_{mnl}^2 - k^2)$ can be approximated for all $(m,n,l) \in \mathscr{A}_\xi$ as

$$\frac{1}{k_{mnl}^2 - k^2} \approx \sum_{q=0}^{p} a_q \frac{k^{2q}}{k_{mnl}^{2q+2}}. \tag{23}$$

Note that choosing all the a_q's to be 1 would be equivalent to a truncated Maclaurin expansion. However, such a choice would not represent an optimal selection of the coefficients for the purpose of minimizing the maximum error of the truncated sum. We fix $\xi = 2$ and truncate (23) at $p = 2$. We also select $a_0 = 1$ to recover the exact solution when $k \to 0$. The remaining coefficients are $a_1 = 0.957641$ and $a_2 = 1.486178$; these were numerically determined to minimize the maximum error in the right-hand side of (23) for k between 0 and $k_{\max}$. For these choices, the maximum relative error is less than 0.1 percent. The Kummer's transformation is now applied to (16):

$$\Sigma_s = \sum_{\mathscr{A}_0}\left\{\cos(k_y y_s)\left(\frac{1}{k_{mnl}^2 - k^2} - \sum_{q=0}^{p} a_q \frac{k^{2q}}{k_{mnl}^{2q+2}}\right)\cdot e^{j(k_x x_s + k_z z_s)}\right\} + \sum_{q=0}^{p} a_q k^{2q} S_q(\overline{R}_s)$$

$$\approx \sum_{\overline{\mathscr{A}}_\xi}\left\{\cos(k_y y_s)\left(\frac{1}{k_{mnl}^2 - k^2} - \sum_{q=0}^{p} a_q \frac{k^{2q}}{k_{mnl}^{2q+2}}\right)\cdot e^{j(k_x x_s + k_z z_s)}\right\} + \sum_{q=0}^{p} a_q k^{2q} S_q(\overline{R}_s) \quad (24)$$

where

$$\overline{R}_s = (x_s, y_s, z_s) \quad (25)$$

and

$$S_q(\overline{R}_s) = \sum_{n=0}^{\infty} \cos(k_y y_s) \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{e^{j(k_x x_s + k_z z_s)}}{k_{mnl}^{2q+2}} \quad (26)$$

is defined for $q = 0, 1, 2$. It should be emphasized that the sum over $\overline{\mathscr{A}}_\xi$ in (24) is finite. In fact, for thin cavities, $\overline{\mathscr{A}}_\xi$ may actually be empty, as in the example $\epsilon_r = 2.2$, $f_{\max} = 15$ GHz, and $h = 1/16$ inch (0.159 cm). Thus, our strategy will be successful if S_q can be efficiently evaluated.

The convergence of S_q will be accelerated by applying a Poisson transformation first to the summation indices m and l. The summand

$$f_{qns}(m,l) = \frac{e^{j(k_x x_s + k_z z_s)}}{k_{mnl}^{2q+2}} \quad (27)$$

is a smooth, slowly decaying function of its arguments (which are now considered to be *continuous* variables defined over the range $(-\infty, \infty)$), implying that its Fourier transform, defined by

$$F_{qns}(\phi, \eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{qns}(m,l)\, e^{j(\phi m + \eta l)}\, dm\, dl \quad (28)$$

is a highly peaked, rapidly decaying function of its arguments. Evaluation of the Fourier transform is accomplished in the Appendix. The Poisson summation formula then states that

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{qns}(m,l) = \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} F_{qns}(2\pi m, 2\pi l). \quad (29)$$

Putting (29) into (26), we find that

$$S_q(\overline{R}_s) = \frac{aw}{q!2^q\pi} \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \rho_{sml}^q e^{-jm\beta w} \cdot \sum_{n=0}^{\infty} \cos(k_y y_s) k_y^{-q} K_q(k_y \rho_{sml}) \quad (30)$$

where

$$\rho_{sml} = \sqrt{(x_s + mw)^2 + (z_s + 2la)^2} \quad (31)$$

and K_q is the modified Bessel function of order q.

Most of the terms in (30) can be neglected without serious error, due to the rapid exponential decay of K_q. We ignore terms for which $\rho_{sml} \geqslant 100h$ and directly sum those terms for which $h/2 \leqslant \rho_{sml} < 100h$, in which region the sum converges rapidly. However, for $\rho_{sml} < h/2$, the sum over n in (30) is slowly convergent and actually diverges as $\rho_{sml} \to 0$. This is not unexpected, since the potentials must exhibit the same singularity as the free-space Green's functions, a singularity proportional to $|\bar{r} - \bar{r}'|^{-1}$.

It is possible to accelerate the convergence of the sums

$$S_{mlq}(\overline{R}_s) = \sum_{n=0}^{\infty} k_y^{-q} \cos(k_y y_s) K_q(k_y \rho_{sml}) = \frac{1}{2}\sum_{n=-\infty}^{\infty} |k_y|^{-q} \cos(k_y y_s) K_q(|k_y| \rho_{sml}) \quad (32)$$

for small values of ρ_{sml}/h and extract the free-space singularity as an explicit term. This is accomplished by yet another application of the Poisson summation formula. Beginning with $q = 0$, we define the Fourier transform

$$G_{ml0s}(\lambda) = \int_{-\infty}^{\infty} \cos(k_y y_s) K_0(|k_y| \rho_{sml}) e^{j\lambda n}\, dn. \quad (33)$$

The transform is evaluated with the aid of [9, eq. (6.699-4)], which results in the following formula for S_{ml0}:

$$S_{ml0}(\overline{R}_s) = \frac{1}{2}\sum_{n=-\infty}^{\infty} G_{ml0s}(2\pi n) = \frac{h}{2}\left[\frac{1}{R_{sm0l}} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{R_{smnl}} + \frac{1}{R_{sm(-n)l}}\right)\right] \quad (34)$$

with

$$R_{smnl} = \sqrt{(y_s + 2nh)^2 + \rho_{sml}^2}. \quad (35)$$

We see that for $m = n = l = 0$ and $s = 4$, the first term in brackets in (34) is the free-space singularity. That this term can be explicitly extracted from the Green's functions is very desirable for application to moment-method problems, where the self-term contribution of the singularity can be integrated in closed form [10]. When the sums are combined for $s=1$ and $s=3$ and for $s=2$ and $s=4$, they represent the multiple-image expansion of the field due to a static charge in a parallel-plate waveguide, one with a magnetic wall and an electric wall. (The static "magnetic wall" is one for which the normal component of the electric field vanishes.) These series can actually be summed explicitly in terms of the Digamma function [11], although we choose a different method of accelerating these sums, as described below.

Proceeding now to the case $q=1$, we encounter a difficulty in evaluating the Fourier transform due to the nonintegrable singularity at $n=-1/2$ in the function

$$\frac{1}{|k_y|}\cos(k_y y_s)K_1(|k_y|\rho_{sml}).$$

In order to proceed, we define the well-behaved auxiliary function

$$\mathscr{S}_{ml1}(\overline{R}_s,\alpha)=\frac{h}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\eta_n}\cos(k_y y_s)K_1\left(\frac{\pi}{h}\eta_n\rho_{sml}\right) \tag{36}$$

with

$$\eta_n(\alpha)=\sqrt{\left(n+\tfrac{1}{2}\right)^2+\alpha^2},\qquad \alpha>0. \tag{37}$$

Then

$$S_{ml1}(\overline{R}_s)=\lim_{\alpha\to 0}\mathscr{S}_{ml1}(\overline{R}_s,\alpha) \tag{38}$$

and we require the Fourier transform of the function

$$g_{ml1s\alpha}(n)=\frac{1}{\eta_n}\cos(k_y y_s)K_1\left(\frac{\pi}{h}\eta_n\rho_{sml}\right). \tag{39}$$

Evaluating the required integral with the aid of [9, eq. (6.726-4)] yields

$$\mathscr{S}_{ml1}(\overline{R}_s,\alpha)=\frac{h}{2\nu\rho_{sml}}\sum_{n=-\infty}^{\infty}(-1)^n e^{-\nu R_{smnl}} \tag{40}$$

with $\nu=\alpha\pi/h$. To facilitate taking the limit of (40) as $\nu\to 0$, we expand R_{smnl} as follows:

$$R_{smnl}=2|n|h+y_s\operatorname{sgn}(n)+O(n^{-1}) \tag{41}$$

where sgn is the signum function and O is the Landau symbol. We now define

$$\chi_{smnl}=R_{smnl}-2|n|h-y_s\operatorname{sgn}(n),\qquad |n|>0 \tag{42}$$

and write

$$\begin{aligned}e^{-\nu R_{smnl}}&=e^{-2h|n|\nu}e^{-\nu y_s\operatorname{sgn}(n)}e^{-\nu\chi_{smnl}}\\&=e^{-2h|n|\nu}e^{-\nu y_s\operatorname{sgn}(n)}\left[1-\nu\chi_{smnl}+O(\nu^2)\right].\end{aligned} \tag{43}$$

By using the expansion (43) in (40), we can collect coefficients of nonvanishing powers of ν and then perform the indicated limit. The result is (hereafter, summation symbols lacking limits are understood to be over the limits $n=1$ to $n=\infty$)

$$S_{ml1}(\overline{R}_s)=\frac{h}{2\rho_{sml}}\cdot\left[h-R_{sm0l}-\sum(-1)^n(\chi_{sm(-n)l}+\chi_{smnl})\right]. \tag{44}$$

A similar derivation yields

$$\begin{aligned}S_{ml2}(\overline{R}_s)=\frac{h}{2\rho_{sml}^2}\Big[&\frac{1}{3}R_{sm0l}^3-hy_s^2+\frac{2}{3}h^3-\frac{1}{2}h\rho_{sml}^2\\&+\frac{1}{3}\sum(-1)^n\left(\chi_{sm(-n)l}^3+\chi_{smnl}^3\right)\\&+2h\sum(-1)^n n\left(\chi_{sm(-n)l}^2+\chi_{smnl}^2\right)\\&-y_s\sum(-1)^n\left(\chi_{sm(-n)l}^2-\chi_{smnl}^2\right)\\&+y_s^2\sum(-1)^n(\chi_{sm(-n)l}+\chi_{smnl})\\&-4hy_s\sum(-1)^n n(\chi_{sm(-n)l}-\chi_{smnl})\\&+4h^2\sum(-1)^n n^2(W_{sm(-n)l}+W_{smnl})\Big]\end{aligned} \tag{45}$$

where W_{smnl} is defined by

$$W_{smnl}=\chi_{smnl}-\frac{\rho_{sml}^2}{4|n|h}+\frac{y_s\rho_{sml}^2}{8n^2h^2}\operatorname{sgn}(n). \tag{46}$$

The alternating sums appearing in (34), (44), and (45) are accelerated by means of a Kummer's transformation. This is demonstrated here for S_{ml0} only. The summand in (34) is expanded in a Maclaurin series in the variable $1/n$, yielding

$$\begin{aligned}\frac{1}{R_{smnl}}+\frac{1}{R_{sm(-n)l}}&=(nh)^{-1}\\&+\frac{1}{8}(2y_s^2-\rho_{sml}^2)(nh)^{-3}+O(n^{-5}).\end{aligned} \tag{47}$$

The accelerated sum is then

$$\begin{aligned}S_{ml0}(\overline{R}_s)=\frac{h}{2}\Bigg[&\frac{1}{R_{sm0l}}-\eta(1)h^{-1}-\frac{1}{8}\eta(3)(2y_s^2-\rho_{sml}^2)h^{-3}\\&+\sum(-1)^n\left\{\frac{1}{R_{smnl}}+\frac{1}{R_{sm(-n)l}}-(nh)^{-1}\right.\\&\left.-\frac{1}{8}(2y_s^2-\rho_{sml}^2)(nh)^{-3}\right\}\Bigg]\end{aligned} \tag{48}$$

where

$$\eta(k)=\sum(-1)^{n+1}k^{-n} \tag{49}$$

is a sum related to the Riemann zeta function and is tabulated in [12]. Similar techniques are used to accelerate the remaining sums.

III. Conclusions

An efficient technique for evaluating the cavity potential Green's functions has been described. It is efficient for two reasons. First, its frequency dependence is represented by the contribution due to a few (often zero) lower order modes of the cavity plus that due to what amounts to three quasi-static terms ($q=0,1,2$). Once the coefficients of the quasi-static terms have been determined, the Green's function can be computed for many different frequencies with little additional expense. In this sense, we can call the expansion broad band. Secondly, although the spectral representations of the coefficients of the quasi-static terms are very slowly converging, we have accelerated their convergence so that even these can be efficiently computed. The singular part has been extracted as an explicit term, which makes the formulation attractive for use in situations where the singularity must be integrated in closed form.

Appendix
The Fourier Transform of $f_{qns}(m,l)$

The integral to be evaluated is

$$F_{qns}(\phi,\eta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{j(k_x x_s+k_z z_s)}}{\left(k_x^2+k_y^2+k_z^2\right)^{q+1}}e^{j(\phi m+\eta l)}\,dm\,dl$$
$$=\frac{aw}{2\pi^2}e^{-j(\phi\beta w/2\pi)}\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{j[(x_s+(w/2\pi)\phi)u+(z_s+(a/\pi)\eta)v]}}{\left(u^2+v^2+k_y^2\right)^{q+1}}\,du\,dv. \tag{A1}$$

We make the substitution

$$u=r\cos\theta \quad v=r\sin\theta \tag{A2}$$

and define $\rho_s(\phi,\eta)$ and $\psi_s(\phi,\eta)$ by

$$x_s+\frac{w}{2\pi}\phi=\rho_s\cos\psi_s \tag{A3}$$

$$z_s+\frac{a}{\pi}\eta=\rho_s\sin\psi_s. \tag{A4}$$

Then

$$F_{qns}(\phi,\eta)=\frac{aw}{2\pi^2}e^{-j(\phi\beta w/2\pi)}\cdot\int_0^{\infty}\frac{2\pi r}{\left(r^2+k_y^2\right)^{q+1}}\left[\frac{1}{\pi}\int_0^{\pi}\cos(r\rho_s\cos\theta)\,d\theta\right]dr. \tag{A5}$$

Recognizing the expression in brackets as an integral representation of the Bessel function of order zero evaluated at $r\rho_s$, we have

$$F_{qns}(\phi,\eta)=\frac{aw}{\pi}e^{-j(\phi\beta w/2\pi)}\int_0^{\infty}\frac{rJ_0(r\rho_s)}{\left(r^2+k_y^2\right)^{q+1}}dr. \tag{A6}$$

The remaining integral can be evaluated using [9, eq. (6.565-4)]. The result is

$$F_{qns}(\phi,\eta)=\frac{aw\rho_s^q}{q!\pi 2^q k_y^q}K_q(k_y\rho_s)e^{-j(\phi\beta w/2\pi)} \tag{A7}$$

which is the desired formula.

References

[1] N. Amitay, V. Galindo, and C. P. Wu, *Theory and Analysis of Phased Array Antennas*. New York: Wiley-Interscience, 1972.

[2] R. F. Harrington, *Field Computation by Moment Methods*. New York: Macmillan, 1968.

[3] D. M. Pozar and D. H. Schaubert, "Analysis of an infinite array of rectangular microstrip patches with idealized probe feeds," *IEEE Trans. Antennas Propagat.*, vol. AP-32, pp. 1101–1107, Oct. 1984.

[4] T. E. Rozzi and W. F. G. Mecklenbrauker, "Wide-band network modeling of interacting inductive irises and steps," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 235–245, Feb. 1975.

[5] L. F. Jelsma, E. D. Tweed, R. L. Phillips, and R. W. Taylor, "Boundary conditions for the four vector potential," *IEEE Trans. Microwave Theory Tech.*, vol. 18, pp. 648–649, Sept. 1970.

[6] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*. New York: McGraw-Hill, 1953.

[7] R. W. Hamming, *Numerical Methods for Scientists and Engineers*. New York: McGraw-Hill, 1961, p. 195.

[8] A Papoulis, *The Fourier Integral and its Applications*. New York: McGraw-Hill, 1962, sec. 3-3.

[9] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*. New York: Academic Press, 1980.

[10] D. R. Wilton *et al.*, "Potential integrals for uniform and linear source distributions on polygonal and polyhedral domains," *IEEE Trans. Antennas Propagat.*, vol. AP-32, pp. 276–281, Mar. 1984.

[11] W. R. Smythe, *Static and Dynamic Electricity*. New York: McGraw-Hill, 1950, p. 203.

[12] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions* (Applied Mathematics Series 55). New York: Dover, 1972, p. 811.

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